

FAMILIES OF CALABI-YAU HYPERSURFACES IN \mathbb{Q} -FANO TORIC VARIETIES

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ABSTRACT. We provide a sufficient condition for a general hypersurface in a \mathbb{Q} -Fano toric variety to be a Calabi-Yau variety in terms of its Newton polytope. Moreover, we define a generalization of the Berglund-Hübsch-Krawitz construction in case the ambient is a \mathbb{Q} -Fano toric variety with torsion free class group and the defining polynomial is not necessarily of Delsarte type. Finally, we introduce a duality between families of Calabi-Yau hypersurfaces which includes both Batyrev and Berglund-Hübsch-Krawitz mirror constructions. This is given in terms of a polar duality between pairs of polytopes $\Delta_1 \subseteq \Delta_2$, where Δ_1 and Δ_2^* are canonical.

INTRODUCTION

A wide class of Calabi-Yau varieties is given by anticanonical hypersurfaces, or more generally complete intersections, in Fano toric varieties. A special interest for such families of Calabi-Yau's arised after the work of Batyrev [5], who defined a duality between the anticanonical linear series of toric Fano varieties which satisfies the requirement of topological mirror symmetry [5,8]:

$$(1) \quad h_{st}^{p,q}(X) = h_{st}^{n-p,q}(X^*), \quad 0 \leq p, q \leq n,$$

where X, X^* are general elements in the dual linear series, $n = \dim(X) = \dim(X^*)$ and $h_{st}^{p,q}$ denote the string-theoretic Hodge numbers. In the case of hypersurfaces, Batyrev mirror construction relies on the polar duality between reflexive polytopes. A different class of Calabi-Yau varieties can be constructed by considering quotients of quasimooth (or transverse) hypersurfaces in weighted projective spaces, i.e. defined by homogeneous polynomials whose affine cone is singular only at the vertex. For such Calabi-Yau varieties, in case they are defined by Delsarte type equations, the physicists Berglund and Hübsch [10] defined a transposition rule for the defining polynomial. The construction has been later refined by Krawitz [33], who introduced the action of finite diagonal symplectic groups. More precisely, the Berglund-Hübsch-Krawitz (BHK for short) transposition rule is a correspondence

$$\{W = 0\} \subset \mathbb{P}(w)/\tilde{G} \longleftrightarrow \{W^* = 0\} \subset \mathbb{P}(w^*)/\tilde{G}^*,$$

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where W^*, w^* and \tilde{G}^* are suitably defined transposed versions of the polynomial W , the set of weights w and the group \tilde{G} (see Section 4). Recently Chiodo and Ruan [17] proved that the BHK transposition rule actually satisfies (1) in terms of the Chen-Ruan orbifold cohomology.

The motivation of this work is first to relate the two constructions of Calabi-Yau varieties described above and secondly to define a duality generalizing both Batyrev and BHK duality. A way of unifying the BHK duality with Batyrev-Borisov duality of reflexive Gorenstein cones has been proposed in [12], where the author develops the vertex algebras approach to mirror symmetry for the BHK construction and suggest a common framework for the latter and Batyrev-Borisov duality (see [12, §7]). The present paper makes the previous framework more explicit in the hypersurface case in terms of polytopes and identifies the needed regularity condition to obtain a duality between families of Calabi-Yau varieties.

More precisely, given a \mathbb{Q} -Fano toric variety X with canonical singularities, we consider families of anticanonical hypersurfaces of X with fixed Newton polytope. Called Θ the anticanonical polytope of X and Δ the given Newton polytope, one such family will be denoted by $\mathcal{F}_{\Delta, \Theta^*}$. We prove the following result, where we recall that a canonical polytope is a lattice polytope whose unique lattice interior point is the origin.

Theorem 1. *Let X be a \mathbb{Q} -Fano toric variety with canonical singularities and let $\Delta \subset M_{\mathbb{Q}}$ be a canonical polytope contained in the anticanonical polytope Θ of X . Then a general hypersurface in $\mathcal{F}_{\Delta, \Theta^*}$ is a Calabi-Yau variety.*

This result gives examples of families of Calabi-Yau varieties in dimension ≥ 5 whose general element is not quasismooth and it is not birational to a hypersurface in a toric Fano variety (see Table 2 for some of them).

Theorem 1 also suggests the definition of a duality between families of Calabi-Yau varieties with fixed Newton polytope in \mathbb{Q} -Fano toric varieties. We will say that a pair (Δ_1, Δ_2) of polytopes is a *good pair* if $\Delta_1 \subseteq \Delta_2$ and both Δ_1 and Δ_2^* are canonical. Clearly the polar (Δ_2^*, Δ_1^*) of a good pair is still a good pair. This involution on good pairs produces the duality

$$\mathcal{F}_{\Delta_1, \Delta_2^*} \subseteq |-K_{X_{\Delta_2}}| \longleftrightarrow \mathcal{F}_{\Delta_2^*, \Delta_1} \subseteq |-K_{X_{\Delta_1^*}}|.$$

This coincides with Batyrev duality when $\Delta_1 = \Delta_2$, since canonical polytopes whose polar is canonical are exactly reflexive polytopes.

Moreover, Theorem 1 allows to define a *generalized BHK transposition rule* where the weighted projective space $\mathbb{P}(w)$ is replaced by a \mathbb{Q} -Fano toric variety with torsion free class group and canonical singularities. More precisely, a generalized BHK family $\mathcal{F}(A, \tilde{G})$ in X/\tilde{G} is a family of hypersurfaces of anticanonical degree defined by a matrix A of exponents for the equations and a symplectic group \tilde{G} . We can associate to it a pair of polytopes $\Delta_1 \subseteq \Delta_2$, where Δ_1 is the Newton polytope of a general element in $\mathcal{F}(A, \tilde{G})$ and Δ_2 is the anticanonical polytope of X/\tilde{G} . The following shows that the generalized BHK transposition rule can be seen as a duality between good pairs.

Theorem 2. *Let (Δ_1, Δ_2) be the pair associated to a generalized Berglund-Hübsch-Krawitz family $\mathcal{F}(A, \tilde{G})$. Then (Δ_1, Δ_2) is a good pair and the pair associated to $\mathcal{F}(A^T, G^*)$ is (Δ_2^*, Δ_1^*) .*

The theorem relies on a toric description of the generalized BHK construction, which had been given in the classical case in [42, §2] and generalizes [42, Proposition 2.9].

The paper is organized as follows. In Section 1 we recall some definitions and basic results about toric varieties and polytopes. In Section 2 we study hypersurfaces in toric varieties and we describe their regularity properties according to the Newton polytope. In Section 3 we prove Theorem 1 and we define the duality between good pairs. Section 4 is devoted to the definition of the generalized BHK mirror construction and to the proof of Theorem 2. Finally Section 5.1 contains some remarks about the stringy Hodge numbers of our families.

1. TORIC BACKGROUND

1.1. Toric varieties. We start recalling some standard facts in toric geometry, see for example [20]. Let N denote a lattice and let $M = \text{Hom}(N, \mathbb{Z})$ be its dual. Let Δ be a polytope in $M_{\mathbb{Q}}$, i.e. the convex hull of a finite subset of $M_{\mathbb{Q}}$. The *polar* of Δ is the polyhedron

$$\Delta^* = \{y \in N_{\mathbb{Q}} : \langle x, y \rangle \geq -1, \forall x \in \Delta\},$$

which clearly contains the origin in its interior and whose facets are contained in the affine hyperplanes of equation $\langle y, v_i \rangle = -1$, where v_i is a vertex of Δ .

It is well known that to any polytope Δ as above one can associate a toric variety $X = X_{\Delta}$ together with a \mathbb{Q} -divisor D . The variety X is the toric variety associated to the normal fan Σ_{Δ} to Δ . If n_1, \dots, n_r are the primitive generators of the one-dimensional cones of Σ_{Δ} and D_1, \dots, D_r are the corresponding integral torus-invariant divisors, then $D = -\sum_i h_{\Delta}(n_i)D_i$, where h_{Δ} is the strictly upper convex function

$$h_{\Delta} : N_{\mathbb{Q}} \rightarrow \mathbb{Q}, \quad h_{\Delta}(y) = \min_{x \in \Delta} \{\langle x, y \rangle\}.$$

Now let $P : \mathbb{Z}^r \rightarrow N$ be the homomorphism defined by $P(e_i) = n_i$, which will be called *P-morphism* of the toric variety. We denote by P^T its transpose and by Q the homomorphism defined by the following exact sequence:

$$0 \longrightarrow M \xrightarrow{P^T} \mathbb{Z}^r \xrightarrow{Q} K \longrightarrow 0,$$

where K is isomorphic to the Class group of X .

The *Cox ring* $\mathcal{R}(X)$ is the polynomial ring $\mathbb{C}[T_1, \dots, T_r]$, where T_i is the defining element of the divisor D_i , graded by K : $\deg(T_i) = Q(e_i)$. Let $\bar{X} = \text{Spec } \mathcal{R}(X) \cong \mathbb{C}^r$. By Cox's construction [20, Theorem 5.1.11] a toric variety X associated to a fan Σ can be described as a GIT-quotient of a quasitorus

$$X \cong (\bar{X} \setminus V(I)) // G_{\Sigma},$$

where $V(I) = V(x^{\hat{\sigma}} \mid \sigma \in \Sigma)$ is the irrelevant locus, that is the subvariety of \bar{X} defined by the vanishing of every monomial of the form $x^{\hat{\sigma}} = \prod_{\rho \notin \sigma(1)} x_{\rho}$ for $\sigma \in \Sigma$, and

$$G_{\Sigma} = \text{Hom}(\text{Cl}(X), \mathbb{C}^*) \simeq \text{Spec}(\mathbb{C}[\text{Cl}(X)]).$$

The big torus $\hat{T} = (\mathbb{C}^*)^r$ naturally acts on the characteristic space $\hat{X} = \bar{X} \setminus V(I)$ by coordinatewise multiplication. Observe that G_{Σ} is the kernel of the natural

homomorphism

$$\Psi : \hat{T} \rightarrow T, \quad (\lambda_1, \dots, \lambda_r) \mapsto (u \mapsto \prod_{j=1}^r \lambda_j^{\langle u, n_j \rangle}),$$

where we identify T with $\text{Hom}(M, \mathbb{C}^*)$.

We finally recall how quotients of toric varieties by finite subgroups of the torus can be described. Let $X = X_{\Sigma, N}$ be a toric variety associated to a fan $\Sigma \subset N_{\mathbb{Q}}$ and let $N \rightarrow N'$ be a lattice monomorphism with finite cokernel G . The fan Σ in $N'_{\mathbb{Q}} = N_{\mathbb{Q}}$ defines a toric variety X' and the inclusion of lattices induces a morphism of toric varieties $X \rightarrow X'$ which is a good geometric quotient by the action of the group G [20, Proposition 3.3.7].

Lemma 1.1. *Let $X = X_{\Sigma, N}$ be a toric variety associated to a fan $\Sigma \subset N_{\mathbb{Q}}$ with torsion free class group, let $\iota : N \rightarrow N'$ be a lattice monomorphism with finite cokernel G and $\pi : X \rightarrow X'$ be the associated finite quotient. If the primitive generators $n_1, \dots, n_r \in N$ of the rays of the fan of X are primitive in N' , then the homomorphism $\pi^* : \mathcal{R}(X') \rightarrow \mathcal{R}(X)$ can be taken to be the identity and $\text{Cl}(X') \cong \text{Cl}(X) \oplus G$.*

Proof. Let P, P' be the P -morphisms of X, X' respectively. By the primitivity assumption on the n_i 's in N' , we have $P' = \iota \circ P$. Thus we have the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & M' & \xrightarrow{P'^T} & \mathbb{Z}^r & \longrightarrow & K' \longrightarrow 0 \\ & & \downarrow \iota^{\vee} & & \downarrow id & & \downarrow pr_1 \\ 0 & \longrightarrow & M & \xrightarrow{P^T} & \mathbb{Z}^r & \longrightarrow & K \longrightarrow 0 \end{array}$$

Observe that the central vertical arrow describes the map π^* in Cox coordinates. Since K is free, then K' is isomorphic to $K \oplus \ker(pr_1)$. Chasing in the diagram one finds that $\ker(pr_1)$ is isomorphic to $M/M' \cong G$. \square

1.2. Polytopes. We recall that a lattice polytope Δ in $M_{\mathbb{Q}}$ containing the origin in its interior is

- *reflexive* if Δ^* is a lattice polytope;
- *canonical* if the origin is its unique interior lattice point;
- *\mathbb{Q} -Fano* if its vertices are primitive in M .

A reflexive polytope can be defined equivalently as a lattice polytope with the origin in its interior such that the integral distance between any of its facets and the origin is equal to one ([5, Theorem 4.1.6]). Clearly with our definition Δ is reflexive if and only if Δ^* is reflexive. A canonical polytope is clearly \mathbb{Q} -Fano since, given a non-primitive vertex mv , $m \in \mathbb{Z}_{>0}$, the vector v would be a non zero interior lattice point. Thus we have the following implications:

$$\text{reflexive} \Rightarrow \text{canonical} \Rightarrow \mathbb{Q}\text{-Fano}.$$

Remark 1.2. In [27] Kasprzyk provided a classification of three dimensional canonical polytopes, available in the Graded Ring Database <http://www.grdb.co.uk/>, and described an approach to the classification in higher dimension. Polytopes with the origin in their interior, sometimes called polytopes with the IP property in the literature, had a key role in the classification of reflexive polytopes of dimension ≤ 4 by Kreuzer and Skarke [35–37].

We recall that a projective normal variety X is \mathbb{Q} -*Fano* if $-K_X$ is \mathbb{Q} -Cartier and ample (in particular it is \mathbb{Q} -Gorenstein) and *Fano* if moreover $-K_X$ is Cartier (in particular it is Gorenstein). Moreover, the following holds (see [20, Theorem 6.2.1, Proposition 11.4.12, Theorem 8.3.4]).

Theorem 1.3. *Let $\Delta \subset N_{\mathbb{Q}}$ be a lattice polytope containing the origin in its interior. Then X_{Δ^*} is*

- \mathbb{Q} -*Fano if and only if Δ is \mathbb{Q} -Fano;*
- \mathbb{Q} -*Fano with canonical singularities if and only if Δ is canonical;*
- *Fano if and only if Δ (or Δ^*) is reflexive.*

We finally introduce the concept of good pair, a key word in the paper.

Definition 1.4. Let Δ_1, Δ_2 be two polytopes in $M_{\mathbb{Q}}$. We will say that (Δ_1, Δ_2) is a *good pair* if $\Delta_1 \subseteq \Delta_2$, and Δ_1, Δ_2^* are canonical (in particular Δ_1 and Δ_2^* are both lattice polytopes).

The following shows that a lattice polytope containing the origin in its interior is canonical as soon as its dual is big enough.

Lemma 1.5. *Let $\Delta \subset M_{\mathbb{Q}}$ be a lattice polytope containing the origin as an interior point. Then the origin is the only lattice interior point of Δ^* .*

Proof. By definition of the polar polytope it is clear that it contains the origin in its interior. Suppose now that Δ^* contains a lattice point $n \neq 0$ in its interior. In particular for $\varepsilon > 0$ small enough $(1 + \varepsilon)n$ is contained in Δ^* . By definition of the polar polytope, this implies that $\langle u, (1 + \varepsilon)n \rangle \geq -1$ for all $u \in \Delta$. Thus every lattice point u in Δ satisfies

$$\langle u, n \rangle \geq \frac{-1}{(1 + \varepsilon)} > -1$$

and hence $\langle u, n \rangle \geq 0$ since $\langle u, n \rangle$ is an integer. It follows that the lattice polytope Δ is contained in the half space $H_n = \{u \in M_{\mathbb{R}} \mid \langle u, n \rangle \geq 0\}$, contradicting the fact that Δ contains the origin as an interior point. \square

As an immediate consequence we have:

Corollary 1.6. *Let Δ_1 and Δ_2 two polytopes in $M_{\mathbb{Q}}$ with $\Delta_1 \subseteq \Delta_2$. Then (Δ_1, Δ_2) is a good pair if and only if Δ_1 and Δ_2^* are lattice polytopes containing the origin as an interior point.*

This also implies the following result, explaining how good pairs behave when changing the underlying lattice up to finite index.

Lemma 1.7. *Let (Δ_1, Δ_2) be a good pair in $M_{\mathbb{Q}}$ and let $M' \subset M$ be an inclusion of lattices with finite index. If the vertices of Δ_1 belong to M' , then (Δ_1, Δ_2) is a good pair in $M'_{\mathbb{Q}}$.*

2. ANTICANONICAL HYPERSURFACES

Let X be a projective toric variety defined by a fan $\Sigma \subset N_{\mathbb{Q}}$ and let $n_1, \dots, n_r \in N$ be the primitive generators of the one dimensional cones of Σ . The *anticanonical polytope* of X is the polytope

$$\Theta = \{m \in M_{\mathbb{Q}} : \langle m, n_i \rangle \geq -1, \forall i\}.$$

The lattice points of Θ naturally give a basis for the Riemann-Roch space of the divisor $-K_X = \sum_i D_i$. In fact, given $u \in \Theta \cap M$, the vector $P^T(u) + \mathbf{1} \in \mathbb{Z}^r$, where $\mathbf{1}$ is the vector with all entries equal to 1, is the vector of exponents of a monomial m_u in the Cox ring $\mathcal{R}(X)$ of degree $[-K_X]$. Conversely, any such monomial can be obtained in the same way.

Given a hypersurface D of X of degree $[-K_X]$ defined by $f = 0$ in Cox coordinates, we define the *support* $\text{supp}(f)$ to be the set of $u \in M$ such that m_u is a monomial of f . Moreover we define the *Newton polytope* of D as the convex hull of the points in its support. Given a lattice polytope Δ contained in Θ we will denote by $\mathcal{F}_{\Delta, \Theta^*}$ the family of anticanonical hypersurfaces of X whose Newton polytope is equal to Δ .

2.1. Regularity of hypersurfaces. In this section we will translate some basic regularity properties of hypersurfaces in $\mathcal{F}_{\Delta, \Theta^*}$ in terms of geometric properties of Δ . We recall that a hypersurface D of a projective toric variety X is called *well-formed* if

$$\text{codim}_D(D \cap \text{Sing}(X)) \geq 2,$$

where $\text{Sing}(X)$ is the singular locus of X .

Example 2.1. In case X is a normalized weighted projective space, i.e. $X = \mathbb{P}(w_1, \dots, w_n)$ with $\text{gcd}(w_1, \dots, \hat{w}_i, \dots, w_n) = 1$, it is known [26] that the general anticanonical hypersurface is well-formed if and only if

$$\text{gcd}(w_1, \dots, \hat{w}_i, \dots, \hat{w}_j, \dots, w_n) \mid \sum_k w_k.$$

We will need the following result, where $D_I := \cap_{i \in I} D_i$ for $I \subseteq \{1, \dots, r\}$.

Lemma 2.2. *Let X be a \mathbb{Q} -Fano toric variety with canonical singularities and let Δ be a lattice polytope contained in its anticanonical polytope. Then D_I is not empty if and only if $\{n_i : i \in I\}$ is contained in a facet of Θ^* and a hypersurface in $\mathcal{F}_{\Delta, \Theta^*}$ contains D_I if and only if $\{n_i : i \in I\}$ is not contained in a facet of Δ^* .*

Proof. Let Θ be the anticanonical polytope of X . By the assumption on X , a fan Σ for X is given by the cones over the facets of Θ^* and the n_i are the vertices of Θ^* . The stratum D_I is not empty if and only if the set $\{n_i : i \in I\}$ is contained in a cone of Σ , or equivalently if the n_i are contained in a facet of Θ^* . This gives the first statement.

Let $u \in \Delta \cap M$, and m_u be the corresponding monomial in homogeneous coordinates. The zero set of the monomial m_u is given by

$$\text{div}(m_u) = \sum_{i=1}^r (\langle u, n_i \rangle + 1) D_i.$$

A hypersurface $f = 0$ in $\mathcal{F}_{\Delta, \Theta^*}$ contains D_I if and only if any monomial m_u in f vanishes along some of the D_i 's with $i \in I$. This means that for all $u \in \text{supp}(f)$ there exists some $i \in I$ with $\langle u, n_i \rangle > -1$. In particular this holds for the vertices of Δ , which means that the n_i 's do not all belong to a single facet of Δ^* . \square

Remark 2.3. In the following results we will usually ask D to be a *general element* in $\mathcal{F}_{\Delta, \Theta^*}$. This means that $D = \{f = 0\}$, where the Newton polytope of D is Δ and the coefficients of f are general. Observe that the support of f is not necessarily equal to $\Delta \cap M$.

Proposition 2.4. *Let X be a \mathbb{Q} -Fano toric variety with canonical singularities and let $\Delta \subset M_{\mathbb{Q}}$ be a lattice polytope contained in the anticanonical polytope Θ of X . A general hypersurface in $\mathcal{F}_{\Delta, \Theta^*}$ is:*

- i) irreducible if and only if n_i belongs to the boundary of Δ^* for any i ;
- ii) well-formed if and only if, anytime n_i, n_j belong to a facet of Θ^* and not to a facet of Δ^* , the segment joining them doesn't contain any lattice point;
- iii) normal if, anytime n_i, n_j belong to a facet of Θ^* and not to a facet of Δ^* , $n_i + n_j$ is not in the interior of Δ^* .

Proof. Since X is \mathbb{Q} -Fano with canonical singularities, the n_i 's are the vertices of Θ^* and the origin is the only interior lattice point of both Θ and Θ^* . In what follows D denotes a general element in $\mathcal{F}_{\Delta, \Theta^*}$.

By the second Bertini's theorem D is reducible if and only if it contains one of the integral invariant divisors D_i for the torus action as a component. Thus i) follows from Lemma 2.2.

By the same Lemma, D contains the stratum D_{ij} if and only if n_i, n_j are contained in a facet of Θ^* and not in a facet of Δ^* . Moreover, X is singular along D_{ij} if and only if the triangle $0, n_i, n_j$ contains a lattice point n outside its vertices. Since the only interior lattice point of Θ^* is the origin, this means that n belongs to the segment between n_i, n_j . This gives ii).

Let $p : \hat{X} \rightarrow X$ be the characteristic space of X , let $\hat{D} = p^{-1}(D)$ and let $\hat{D}_i = p^{-1}(D_i)$. By Serre's criterion [25, Proposition 8.23, Ch.II] \hat{D} is normal if and only if it is smooth in codimension one. By the first Bertini's theorem this happens if and only if $\hat{D}_{ij} := \hat{D}_i \cap \hat{D}_j$ is not contained in the singular locus of \hat{D} , whenever it is not empty. By Lemma 2.2 \hat{D}_{ij} is not empty and it is contained in \hat{D} when n_i, n_j belong to the same facet of Θ^* but not to a facet of Δ^* . Under these conditions, \hat{D} is singular along \hat{D}_{ij} if and only if, for any $u \in \text{supp}(f)$, $(u, n_j) > -1$ whenever $(u, n_i) = 0$, and similarly changing the role of i and j (this is equivalent to ask that the partial derivatives of the equation of \hat{D} in homogeneous coordinates vanish along \hat{D}_{ij}). Since there exists no $u \in \Delta \cap M$ such that $(u, n_i) = (u, n_j) = -1$, this is equivalent to ask that $(u, n_i + n_j) > -1$ for all $u \in \Delta \cap M$, i.e. that $n_i + n_j$ belongs to the interior of Δ^* .

We recall that p is a GIT quotient for the action of a quasi-torus T . The divisor \hat{D} is T -invariant, being defined by a homogeneous polynomial in $\mathcal{R}(X)$. This implies that $p|_{\hat{D}} : \hat{D} \rightarrow D$ is still a GIT quotient for the action of the group T/T_0 , where T_0 is the subgroup of T acting trivially on \hat{D} . Since \hat{D} is normal, it follows that D is normal (see for example [20, Lemma 5.0.4]). This proves iii). \square

2.2. Hypersurfaces with canonical singularities. Let X be a \mathbb{Q} -Gorenstein normal variety over \mathbb{C} of dimension ≥ 2 and let D be a \mathbb{Q} -Cartier divisor on X . Given a resolution $f : \tilde{X} \rightarrow X$, that is a proper birational morphism such that \tilde{X} is smooth, one can write

$$(K_{\tilde{X}} + f_*^{-1}D) - f^*(K_X + D) \equiv \sum_{i=1}^r a(X, D, E_i)E_i,$$

where E_1, \dots, E_r are the distinct irreducible components of the exceptional divisor of f and $f_*^{-1}D$ denotes the proper birational transform of D (see [32, Remark 6.6] for the precise meaning of this equation). The *discrepancy* of the pair (X, D) ,

denoted by $\text{discrep}(X, D)$, is the infimum of the values $a(X, D, E)$, as E varies over all exceptional divisors of the resolutions of X . The pair (X, D) is *canonical* if $\text{discrep}(X, D) \geq 0$ and X has *canonical singularities* if $(X, 0)$ is canonical.

In order to compute the discrepancy of a pair (X, D) it is enough to consider the minimum over the values $a(X, D, E)$ as E varies among the exceptional divisors of a given *log resolution* of (X, D) , i.e. a resolution such that $\text{Exc}(f) + f_*^{-1}(D)$ has pure codimension 1 and is a divisor with simple normal crossings [32, Definition 6.21]. Such a resolution always exists by a theorem of Hironaka [30, Theorem 10.45]. We recall the following result, which relates the discrepancy of a pair (X, D) to the discrepancy of D .

Theorem 2.5 ([28]). *Let X be a normal variety over \mathbb{C} and let D be a normal divisor on X such that $K_X + D$ is \mathbb{Q} -Cartier and $\text{codim}_D(\text{Sing}(X) \cap D) \geq 2$. Then*

$$(2) \quad \text{discrep}(D) \geq \text{discrep}(\text{center} \subset D, X, D),$$

where the right hand side is the infimum of the values $a(X, D, E)$, where $f(E) \subset Z$. In particular D has canonical singularities if the pair (X, D) is canonical.

Proof. The inequality follows from [31, Proposition 5.46] taking $Z = S$ and $B = 0$ or [28, §17.2]. The last statement immediately follows since $\text{discrep}(\text{center} \subset D, X, D) \geq \text{discrep}(X, D)$. \square

Proposition 2.6. *Let X be a \mathbb{Q} -Fano toric variety with canonical singularities and let $\Delta \subset M_{\mathbb{Q}}$ be a lattice polytope contained in the anticanonical polytope Θ of X . If Δ is a canonical polytope then the general element D of $\mathcal{F}_{\Delta, \Theta^*}$ is well-formed, normal and has canonical singularities.*

Proof. If Δ is canonical, then Δ^* is a polytope and its only interior lattice point is the origin by Lemma 1.5. By Proposition 2.4 we have that D is well-formed since, if n_i, n_j belong to a facet of Θ^* and not to a facet of Δ^* , then the segment joining them intersects the boundary of Δ^* only at n_i, n_j . Moreover, by the same proposition, D is normal.

Since D is general, there exists a toric resolution of singularities $f : \tilde{X} \rightarrow X$ which is a log resolution for D , obtained by means of a refinement $\tilde{\Sigma}$ of the fan Σ of X . This can be obtained taking first a toric resolution of the singularities of X and then successive toric blow-ups along the base locus of $\mathcal{F}_{\Delta, \Theta^*}$ until its proper transform is base point free. By the first Bertini's theorem the general element \tilde{D} of such proper transform is smooth. Moreover, the same theorem implies that \tilde{D} intersects transversally each component of the exceptional locus, since its restriction to any such component is base point free.

Let E be an exceptional divisor of f and let $n \in N$ be the primitive generator of the corresponding ray of $\tilde{\Sigma}$. Observe that

$$\text{mult}_E(K_{\tilde{X}} - f^*(K_X)) = -1 - \text{mult}_E f^*(K_X)$$

since E is one of the integral torus invariant divisors of \tilde{X} . We can write $D = \text{div}(\chi) - K_X$, where χ is a general linear combination of χ^u , where u belongs to the support $\text{supp}(\phi)$ of a defining equation ϕ of D . Then

$$\begin{aligned} \text{mult}_E(f^*(D) - f_*^{-1}(D)) &= \text{mult}_E(f^*(D)) = \text{mult}_E(f^*\text{div}(\chi)) - \text{mult}_E f^*(K_X) \\ &= \min_{u \in \text{supp}(\phi)} \{\text{mult}_E(f^*\chi^u)\} - \text{mult}_E f^*(K_X) = \min_{u \in \text{supp}(\phi)} \{(u, n)\} - \text{mult}_E f^*(K_X), \end{aligned}$$

where the third equality is due to the generality assumption on D and the last equality to the fact that

$$\text{mult}_E f^* \text{div}(\chi^u) = \text{mult}_E \text{div}(f^* \chi^u) = (u, n).$$

This gives

$$(3) \quad a(X, D, E) = -1 - \min_{u \in \text{supp}(\phi)} \{(u, n)\} = -1 - \min_{u \in \Delta \cap M} \{(u, n)\},$$

where the second equality is due to the fact that $\text{supp}(\phi)$ contains the vertices of Δ . Such discrepancy is non-negative since Δ^* has no non-zero interior lattice point. Theorem 2.5 thus implies that D has canonical singularities. \square

Corollary 2.7. *Let X be a \mathbb{Q} -Fano toric variety with canonical singularities, Δ be a canonical polytope contained in its anticanonical polytope Θ and D be a general element in $\mathcal{F}_{\Delta, \Theta^*}$. A birational toric morphism $f : \tilde{X} \rightarrow X$ induces a crepant morphism $f^*(D) \rightarrow D$ if and only if the rays of the fan of \tilde{X} are generated by nonzero lattice points in Δ^* .*

Proof. It follows from (3) that $a(X, D, E) = 0$ if and only if $n \in \Delta^*$. \square

Remark 2.8. By the proof of Proposition 2.6 the discrepancies of a general anticanonical hypersurface D only depend on its Newton polytope Δ .

Remark 2.9. As a consequence of the adjunction Conjecture [30, Theorem 4.9] formulated by Shokurov and Kollar, the inequality (3) is actually an equality. We now show that, under such conjecture, the condition on the polytope Δ in Proposition 2.6 is also a necessary condition for D to be normal with canonical singularities. Assume that $n \in N$ is a non-zero primitive vector in the interior of Δ^* . Let $\tilde{\Sigma}$ be a smooth fan refining the star subdivision of the fan Σ of X induced by n . This gives a resolution f of X and n corresponds to an exceptional divisor E of f . Let σ be the cone of Σ containing n in its interior. The primitive generators of the rays of σ are not contained in a facet of Δ^* , since otherwise this would also be a facet of Θ^* and n would be an interior point of Θ^* , contradicting the fact that X has canonical singularities (see Theorem 1.3). Thus $f(E) \subseteq D$ by Lemma 2.2. Since n is in the interior of Δ^* , the computation in the proof of Proposition 2.6 gives that $a(X, D, E) < 0$, thus by [30, Theorem 4.9] D has a non-canonical singularity.

Corollary 2.10. *Let X be a \mathbb{Q} -Fano toric variety with canonical singularities. If the convex hull of the lattice points of the anticanonical polytope of X is a canonical polytope, then a general anticanonical hypersurface of X is well-formed, normal and has canonical singularities.*

Proof. It follows from Proposition 2.6 taking Δ to be the the convex hull of $\Theta \cap M$. \square

2.3. Quasismooth hypersurfaces. A hypersurface D of a projective toric variety X is called *quasismooth* (or *transverse*) if $p^{-1}(D)$ is smooth, where $p : \tilde{X} \rightarrow X$ is the quotient map in the Cox construction of X .

If X is a weighted projective space, a quasismooth hypersurface of X of dimension ≥ 3 is known to be well-formed, unless it is isomorphic to a toric stratum [21, Proposition 6]. This result can be generalized as follows.

Proposition 2.11. *Let X be a projective toric variety whose irrelevant locus has codimension > 4 in \hat{X} . A quasismooth hypersurface D of X is either well-formed or it is isomorphic to a toric stratum of X .*

Proof. Let f be a defining element for D in the Cox ring $R(X) = \mathbb{C}[x_1, \dots, x_r]$. Assume that D is not well-formed, in particular it contains a codimension two toric stratum of X . Thus we can assume f to be of the form

$$f(x_1, \dots, x_r) = x_1 f_1 + x_2 f_2.$$

Computing the partial derivatives of f one can see that they all vanish along the subset S of \hat{X} defined by $\{x_1 = x_2 = f_1 = f_2 = 0\}$. If neither f_1 or f_2 is constant, we have that

$$\dim(S) \geq \dim(\hat{X}) - 4 > \dim(\bar{X} - \hat{X}),$$

contradicting the fact that D is quasismooth. Thus we can assume that f_1 is constant, so that $f(x_1, \dots, x_r) = \alpha x_1 + x_2 f_2$ is isomorphic to the stratum $x_1 = 0$ by the isomorphism $(x_2, \dots, x_r) \mapsto (-x_2 \alpha^{-1} f_2, x_2, \dots, x_r)$. \square

Quasismooth and well-formed anticanonical hypersurfaces give a class of hypersurfaces with canonical singularities. However, as we will observe later, such class is quite small in dimension bigger than three.

Proposition 2.12. *Let D be an anticanonical hypersurface of a projective toric variety X . If D is quasismooth and well-formed, then D has canonical singularities.*

Proof. Since D is quasismooth, then D is normal by the proof of Proposition 2.4. Moreover, by [30, Proposition 4.5, (1) and (5)] the adjunction formula holds for D and gives that $K_D \sim \mathcal{O}_D$. In particular D has Gorenstein singularities. Moreover, since \hat{D} is smooth, the singularities of D are rational [15, Corollaire]. By [29, Corollary 11.13] Gorenstein rational singularities are canonical. \square

Remark 2.13. In [5] Batyrev considered a different notion of regularity: an anticanonical hypersurface Y of a toric Fano variety X is regular if the intersection of Y with any toric stratum of X is either empty or smooth of codimension one. Any regular hypersurface is quasismooth, see [9, Proposition 4.15] or [19, Proposition 5.3].

2.4. Examples. Given a \mathbb{Q} -Fano toric variety with canonical singularities and with anticanonical polytope Θ , we can consider three different properties for the convex hull $\bar{\Theta}$ of the lattice points of Θ : canonical, reflexive and quasismooth. Here we say that $\bar{\Theta}$ is *quasismooth* if such property holds for the general anticanonical hypersurface. In the case of weighted projective spaces it is known that the three concepts are equivalent in dimension 2 and 3. In dimension 4, canonical implies reflexive [43, Theorem, §3]. Moreover, for weighted projective spaces of any dimension, quasismooth implies canonical [43, Lemma 2]. In higher dimension the concepts of reflexive and quasismooth are unrelated and there are examples of canonical polytopes which are neither quasismooth nor reflexive. In Table 1 we show the number of weight systems $w = (w_1, \dots, w_6)$ with $w_i \leq 10$ such that the anticanonical polytope Θ of $\mathbb{P}(w)$ is reflexive (F), Θ is reflexive (R), $\bar{\Theta}$ is canonical (C) and not reflexive and we distinguish whether the general anticanonical hypersurface of $\mathbb{P}(w)$ is quasismooth (Q) or not. The properties of the anticanonical polytope can be checked by means of Magma [14] and the programs available here:

<http://goo.gl/A7W17Z>.

See also the Calabi-Yau data webpage by Kreuzer and Skarke

<http://hep.itp.tuwien.ac.at/~kreuzer/CY/>.

weights up to	F	R	Q and not R	C not R and not Q
2	3	4	1	0
3	6	13	5	2
4	10	39	11	3
5	15	83	30	30
6	28	164	45	63
7	31	300	89	193
8	44	524	133	358
9	52	833	190	747
10	71	1278	269	1221

TABLE 1. Counting weight systems in dimension 5

Remark 2.14. Let X be a \mathbb{Q} -Fano toric variety and assume that there exists a toric Fano variety X' and a birational toric map $X \rightarrow X'$ which induces a bijection between the anticanonical linear series $|-K_X|$ and $|-K_{X'}|$. By standard facts in toric geometry, this map is induced by an isomorphism $\varphi : M \rightarrow M'$ which gives a bijection between the lattice points of the anticanonical polytope Θ of X and those of the anticanonical polytope Θ' of X' . In particular $\varphi_{\mathbb{Q}}$ induces an isomorphism between the convex hulls of the lattice points of Θ and Θ' . Since X' is Fano, the polytope Θ' is reflexive by [5, Theorem 4.1.9], thus the convex hull of the lattice points of Θ is reflexive. This shows that, if $\bar{\Theta}$ is not reflexive, then the general anticanonical hypersurface of X is not (torically) birational to a hypersurface in a toric Fano variety.

Example 2.15 (R , C and Q). In dimension 3 there are 104 weighted projective spaces with canonical singularities; for 95 of them the convex hull of lattice points of the anticanonical polytope is a canonical, reflexive and quasismooth polytope. Moreover, for 14 of these weight systems the anticanonical polytope is reflexive, i.e. the weighted projective space is Fano.

Example 2.16 (R and not Q). $X = \mathbb{P}(1, 1, 1, 3, 4)$ is a toric \mathbb{Q} -Fano variety such that $\bar{\Theta}$ is reflexive. Observe that an anticanonical hypersurface of X is defined by an equation of the form:

$$f(x_1, \dots, x_5) = x_1 f_1 + x_2 f_2 + x_3 f_3 + x_4 f_4,$$

since it has degree 10 and there is no power of such degree in the variable x_5 . All partial derivatives vanish at the point $(0 : 0 : 0 : 0 : 1)$ since f_1, f_2, f_3, f_4 do not contain a power of x_5 . Thus the general anticanonical hypersurface of X is not quasismooth.

Example 2.17 (Q and not R). $X = \mathbb{P}(1, 1, 1, 1, 1, 2)$ is a toric \mathbb{Q} -Fano variety such that $\bar{\Theta}$ is canonical and not reflexive. A general anticanonical hypersurface of X is defined by an equation of the form

$$f(x_1, \dots, x_6) = x_1 f_1 + x_2 f_2 + x_3 f_3 + x_4 f_4 + x_5 f_5.$$

By Bertini's theorem, the singular locus of f in \mathbb{C}^6 is contained in the base locus of the corresponding linear system, which only contains the point $(0 : 0 : 0 : 0 : 0 : 1)$. The partial derivatives of f do not vanish at such point since we can assume that f_1 (for example) contains the monomial x_6^3 . Thus $f = 0$ is quasismooth. More examples are given in Table 2.

Example 2.18 (C not R and not Q). $X = \mathbb{P}(1, 1, 2, 3, 3, 3)$ is a toric \mathbb{Q} -Fano variety such that $\bar{\Theta}$ is canonical and not reflexive and such that the general anticanonical hypersurface is not quasismooth. More examples are given in Table 2.

$C, \text{not } R, Q$	$C, \text{not } R, \text{not } Q$
$(1, 1, 1, 1, 1, 2)$	$(1, 1, 2, 3, 3, 3)$
$(1, 2, 2, 2, 3, 4)$	$(1, 1, 2, 3, 3, 4)$
$(1, 1, 1, 1, 2, 3)$	$(1, 1, 1, 2, 3, 3)$
$(1, 1, 2, 2, 2, 3)$	
$(1, 2, 3, 3, 3, 3)$	
$(1, 1, 1, 3, 3, 4)$	
$(1, 2, 2, 3, 3, 4)$	
$(1, 1, 2, 2, 3, 4)$	
$(1, 1, 3, 3, 3, 4)$	
$(1, 2, 2, 3, 4, 4)$	
$(1, 1, 1, 2, 2, 3)$	

TABLE 2. Weights ≤ 4 having quasismooth and non-quasismooth general anticanonical hypersurface

3. A DUALITY BETWEEN FAMILIES OF CALABI-YAU HYPERSURFACES

We recall that an n -dimensional normal projective variety Y is a *Calabi-Yau variety* if it has canonical singularities, $K_Y \cong \mathcal{O}_Y$ and $h^i(Y, \mathcal{O}_Y) = 0$ for $0 < i < n$.

In [5, Theorem 4.1.9] Batyrev proved that a projective toric variety X_Δ is Fano, or equivalently its anticanonical polytope Δ is reflexive, if and only if regular anticanonical hypersurfaces Y of X are Calabi-Yau varieties. Moreover he defines a duality between families of anticanonical hypersurfaces of Fano toric varieties:

$$\mathcal{F}_{\Delta, \Delta^*} \subseteq |-K_{X_\Delta}| \longleftrightarrow \mathcal{F}_{\Delta^*, \Delta} \subseteq |-K_{X_{\Delta^*}}|.$$

In this section we will introduce a generalization of this duality in case X is \mathbb{Q} -Fano and the family of hypersurfaces is not necessarily the full anticanonical linear system. Such generalization is based on the result given in Theorem 1; using the characterization of Proposition 2.6 we can now prove it. Observe that by Remark 2.9, if the equality holds in (3), this would provide a characterization of \mathbb{Q} -Fano toric varieties whose general anticanonical hypersurfaces are Calabi-Yau.

Proof of Theorem 1. In what follows D will denote a general anticanonical hypersurface of $\mathcal{F}_{\Delta, \Theta^*}$. By Proposition 2.6 D is well-formed, normal and has canonical singularities. By [30, Proposition 4.5, (1) and (5)] the adjunction formula holds for D , giving that K_D is trivial. Moreover we have the exact sequence

$$0 \longrightarrow \mathcal{O}_X(K_X) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0,$$

which induces the exact sequence

$$\dots \longrightarrow H^i(X, \mathcal{O}_X) \longrightarrow H^i(D, \mathcal{O}_D) \longrightarrow H^{i+1}(X, \mathcal{O}_X(K_X)) \longrightarrow \dots$$

Since $h^i(X, \mathcal{O}_X) = 0$ for $i > 0$ and $h^{i+1}(X, \mathcal{O}_X(K_X)) = h^{\dim(X)-i-1}(X, \mathcal{O}_X)$ for $i < \dim(X)-1$ by Serre-Grothendieck duality, we obtain the vanishing of $h^i(D, \mathcal{O}_D)$ for $0 < i < \dim(D)$. Thus D is a Calabi-Yau variety. \square

Remark 3.1. In [39, Theorem 2.25] the author states that a general anticanonical hypersurface of a projective toric variety is a Calabi-Yau variety if and only if the convex hull of the lattice points of its anticanonical polytope is reflexive. This is not true in general, see Example 2.17.

Observe that a good pair of polytopes $\Delta_1 \subset \Delta_2$ naturally produces a family of Calabi-Yau varieties in a \mathbb{Q} -Fano projective toric variety. In fact, the toric variety $X := X_{\Delta_2}$ defined by the normal fan to Δ_2 is \mathbb{Q} -Fano with canonical singularities by Theorem 1.3 and Δ_2 is its anticanonical polytope. The subpolytope $\Delta_1 \subseteq \Delta_2$ identifies the family $\mathcal{F}_{\Delta_1, \Delta_2^*}$ of anticanonical hypersurfaces of X , whose general element is a Calabi-Yau variety by Proposition 2.6. By our definition of good pair we immediately have that if (Δ_1, Δ_2) is a good pair in $M_{\mathbb{Q}}$, then its polar (Δ_2^*, Δ_1^*) is a good pair in $N_{\mathbb{Q}}$. This provides a duality between families of Calabi-Yau hypersurfaces of \mathbb{Q} -Fano toric varieties:

$$\mathcal{F}_{\Delta_1, \Delta_2^*} \subseteq |-K_{X_{\Delta_2}}| \longleftrightarrow \mathcal{F}_{\Delta_2^*, \Delta_1} \subseteq |-K_{X_{\Delta_1^*}}|.$$

Proposition 3.2. *If $\Delta_1 = \Delta_2$, then the duality between good pairs is Batyrev duality.*

Proof. If $\Delta_1 = \Delta_2$, then Δ_2 and Δ_2^* are lattice polytopes, thus Δ_2 is reflexive. By Theorem 1.3 this means that X_{Δ_2} is a Fano variety. Moreover $\mathcal{F}_{\Delta_2, \Delta_2^*}$ is the family of all anticanonical hypersurfaces of X_{Δ_2} . \square

Example 3.3. Let $X = \mathbb{P}^2$ and let Δ_2 be its anticanonical polytope. Let Δ_1 be the lattice polytope whose vertices are $(1, 0), (0, 1), (-1, 0), (0, -1)$. The pair (Δ_1, Δ_2) is a good pair and the toric variety $Y := X_{\Delta_1^*}$ is $\mathbb{P}^1 \times \mathbb{P}^1$. The family $\mathcal{F}_{\Delta_1, \Delta_2^*}$ in X is given by $\alpha_1 x_1^2 x_2 + \alpha_2 x_2^2 x_1 + \alpha_3 x_1 x_3^2 + \alpha_4 x_2 x_3^2 + \alpha_5 x_1 x_2 x_3$ where $\alpha_i \in \mathbb{C}^*$, whereas the dual family $\mathcal{F}_{\Delta_2^*, \Delta_1}$ is defined by the equation $\beta_1 y_1^2 y_3 y_4 + \beta_2 y_1 y_2 y_3^2 + \beta_3 y_2^2 y_4^2 + \beta_4 y_1 y_2 y_3 y_4$, $\beta_i \in \mathbb{C}^*$ in homogeneous coordinates of Y (in fact up to rescaling the variables one can put all coefficients β_i to be equal to one).

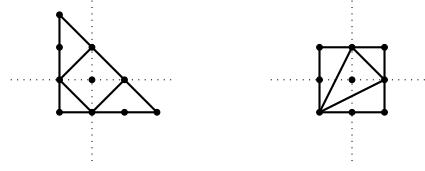


FIGURE 1. The good pairs (Δ_1, Δ_2) and (Δ_2^*, Δ_1^*)

Remark 3.4. Families of Calabi-Yau varieties associated to pairs of nested reflexive polytopes already appeared in the literature in relation with the phenomenon of extremal transition [23, 38, 41]. An extremal transition between two families \mathcal{F} and \mathcal{G} of Calabi-Yau manifolds occurs when there is a degeneration of \mathcal{F} to a singular

family \mathcal{F}_0 whose resolution is isomorphic to \mathcal{G} . An inclusion of reflexive polytopes $\Delta_1 \subset \Delta_2$ can produce such a transition since the family $\mathcal{F}_{\Delta_1, \Delta_2^*}$ is a degeneration of $\mathcal{F}_{\Delta_2, \Delta_2^*}$ (obtained putting some coefficients to zero) and is birational to the family $\mathcal{F}_{\Delta_1, \Delta_1^*}$. Such extremal transitions have been used to prove connectedness of moduli spaces of Calabi-Yau manifolds [3, 4, 11, 16].

Proposition 3.5. *Let (Δ_1, Δ_2) and (Δ'_1, Δ_2) be two good pairs. Then the dual families of $\mathcal{F}_{\Delta_1, \Delta_2^*}$ and $\mathcal{F}_{\Delta'_1, \Delta_2^*}$ are birational.*

Proof. This follows from the fact that the toric varieties $X_{(\Delta'_1)^*}$ and $X_{(\Delta_1)^*}$ are compactifications of the same torus $T_N = \text{Spec } \mathbb{C}[N]$ and that the dual families in T_N are both defined by linear combinations of the monomials corresponding to the points of the polytope Δ_2^* . \square

In the next section we will show that the duality between good pairs also includes Berglund-Hübsch-Krawitz duality. This implies that Proposition 3.5 can be seen as a generalization of [42, Theorem 3.1]. See also [22] for a recent generalization of the proposition in a more general setting and in terms of derived equivalences.

4. BERGLUND-HÜBSCH-KRAWITZ (BHK) DUALITY

4.1. The BHK construction. We will recall a mirror construction due to the physicists Berglund and Hübsch [10] and later refined by Krawitz in [33]. Let $\mathbb{P}(w) = \mathbb{P}(w_1, \dots, w_n)$ be a normalized weighted projective space and let W be a homogeneous polynomial of Delsarte type, i.e. having the same number of monomials and variables. Up to rescaling the variables, we can assume that

$$(4) \quad W(x_1, \dots, x_n) = \sum_{i=1}^n \prod_{j=1}^n x_j^{a_{ij}},$$

so that W is uniquely determined by its matrix of exponents $A = (a_{ij})$. We will denote by Y_W the hypersurface defined by W in $\mathbb{P}(w)$ and we will assume that

- (i) A is invertible over \mathbb{Q} ,
- (ii) Y_W is quasismooth,
- (iii) $\deg(W) = \sum_{i=1}^n w_i$ (*Calabi-Yau condition*).

The assumptions (ii) and (iii) imply that Y_W is a Calabi-Yau variety by Proposition 2.12 and [18, Lemma 1.11].

Remark 4.1. By the proof of [43, Lemma 2] the condition of quasismoothness implies that the matrix A is invertible over \mathbb{Q} . Thus condition (i) in the above construction is redundant. Moreover, it can be easily proved that asking A to be invertible over \mathbb{Q} is equivalent to say that the convex hull of the elements $u_1, \dots, u_n \in M$ corresponding to the monomials of W is a simplex in $M_{\mathbb{R}}$.

If we now consider the transposed matrix of A , this defines in the same way a Delsarte type polynomial W^* . A set of weights $w^* = (w_1^*, \dots, w_n^*)$ which makes W^* homogeneous is given by the smallest integer multiple of the vector

$$q^* = (A^T)^{-1} \cdot \mathbf{1},$$

where $\mathbf{1}$ denotes the column vector with all entries equal to 1. By the quasismoothness assumption it follows that w^* can be chosen with all positive entries (see Remark 4.2). Thus W^* defines a hypersurface Y_{W^*} in $\mathbb{P}(w^*)$. By [34, Theorem 1] W^* is still quasismooth and an easy computation shows that it satisfies the

Calabi-Yau condition in $\mathbb{P}(w^*)$. Thus Y_{W^*} is a Calabi-Yau variety. The Berglund-Hübsch-Krawitz construction gives a duality

$$Y_W/\tilde{G} \longleftrightarrow Y_{W^*}/\tilde{G}^*,$$

where \tilde{G} denotes a quotient group G/J , with $J \subseteq \mathrm{SL}_n(\mathbb{C})$ the subgroup of diagonal automorphisms inducing the identity on $\mathbb{P}(w)$ and G a subgroup of diagonal automorphisms in $\mathrm{SL}_n(\mathbb{C})$ containing J and acting trivially on W , i.e.

$$W(g(x)) = W(x), \quad \forall g \in G.$$

The transposed group \tilde{G}^* is defined as G^*/J^* , where J^* is the analogous of J for $\mathbb{P}(w^*)$ and G^* is defined by

$$(5) \quad G^* := \left\{ \prod_{j=1}^n (\rho_j^*)^{\alpha_j} \mid \prod_{j=1}^n x_j^{\alpha_j} \text{ is } G\text{-invariant} \right\},$$

where $\rho_j^* := \mathrm{diag}(\exp(2\pi i a^{j1}), \dots, \exp(2\pi i a^{jn}))$ and a^{ji} are the entries of A^{-1} . Several equivalent definitions for the transposed group can be found in [1, §3]. The groups \tilde{G} and \tilde{G}^* both act symplectically [1, Proposition 2.3], thus Y_W/\tilde{G} and Y_{W^*}/\tilde{G}^* are both Calabi-Yau varieties. In fact we will prove in Proposition 4.7 that they are anticanonical hypersurfaces of the \mathbb{Q} -Fano toric varieties $\mathbb{P}(w)/\tilde{G}$ and $\mathbb{P}(w^*)/\tilde{G}^*$.

In [17, Theorem 2] Chiodo and Ruan proved that Y_W/\tilde{G} and Y_{W^*}/\tilde{G}^* have symmetric Hodge diamonds for the Chen-Ruan orbifold cohomology.

Remark 4.2. Observe that, by the above definition, we have that

$$A^T q^* = \mathbf{1} \iff \sum_{i=1}^n q_i^* P^T(u_i) = 0 \iff \sum_{i=1}^n q_i^* u_i = 0,$$

where $u_1, \dots, u_n \in M$ are the points corresponding to the monomials of W , i.e. $P^T(u_i) + \mathbf{1}$ is the i -th row of A . Moreover

$$\sum_{i=1}^n q_i^* = \mathbf{1}^T (A^T)^{-1} \mathbf{1} = \mathbf{1} A^{-1} \mathbf{1} = 1.$$

Thus the entries of the vector q^* are the barycentric coordinates of the origin in the simplex with vertices u_1, \dots, u_n . In particular all the entries of q^* are positive if and only if the origin lies in the interior of the simplex. Since X_W is quasismooth, by [43, Lemma 2] the simplex contains the origin in its interior.

4.2. The generalized BHK construction. We now describe a natural generalization of the BHK construction where weighted projective spaces are replaced by their higher class group rank analogs: \mathbb{Q} -Fano toric varieties with torsion-free class group.

Construction 4.3. Let $X = X_\Sigma$ be a \mathbb{Q} -Fano toric variety with torsion free class group. Denote by n_1, \dots, n_r the minimal generators of the rays of Σ and by Δ_2 the anticanonical polytope of X . Let $\Delta_1 \subseteq \Delta_2$ be a lattice polytope containing the origin in its interior and let us denote by u_1, \dots, u_s its vertices.

Note that by Corollary 1.6 (Δ_1, Δ_2) is a good pair and hence u_1, \dots, u_s are primitive lattice vectors. Moreover n_1, \dots, n_r generate the whole lattice N since $\mathrm{Cl}(X)$ is torsion-free. Instead of considering a single polynomial W (made unique

by a rescaling of the variables) as in the original construction, we consider a family of polynomials of the form

$$W(x_1, \dots, x_r) = \sum_{i=1}^s \alpha_i \prod_{j=1}^r x_j^{a_{ij}}, \quad \alpha_i \in \mathbb{C}^*,$$

uniquely determined by the matrix

$$A = (a_{ij})_{\substack{i=1, \dots, s \\ j=1, \dots, r}} \quad \text{with } a_{ij} = \langle u_i, n_j \rangle + 1.$$

By construction W is homogeneous of degree $[-K_X]$ and the equation $W = 0$ defines a hypersurface Y_W of X . Observe that, by the proof of [43, Lemma 2], the quasimoothness condition for W implies that its Newton polytope contains the origin in its interior.

An important fact to notice is that by the hypotheses made on u_1, \dots, u_s , the matrix A determines the variety X .

Lemma 4.4. *Since Δ_1 is a lattice polytope containing the origin in its interior, it is possible to recover the toric variety X from the matrix A .*

Proof. Since the polytopes Δ_1 and Δ_2^* are full dimensional, we know that the matrix $A - \mathbf{1} = (\langle u_i, n_j \rangle)_{i=1, \dots, s, j=1, \dots, r}$ has rank n , so that we may choose n linearly independent rows l_{i_1}, \dots, l_{i_n} of it. These define a submatrix P' , equal to the product UP where the rows of U are the coefficients of u_{i_1}, \dots, u_{i_n} and P is the matrix of the P -morphism of X (see §1).

The lattice $N' = UN$ is a sublattice of N of index $|N/N'| = |\det(U)|$ and by [20, Prop. 3.3.7], the group K' in the short exact sequence

$$(6) \quad 0 \longrightarrow (N')^\vee = M' \xrightarrow{(P')^T} \mathbb{Z}^r \xrightarrow{Q'} K' \longrightarrow 0,$$

is isomorphic to the class group of the geometric quotient X/H where $H = N/N'$. Since $P' = UP$ we have $K' \simeq K \times H$ and we can factorize the sequence as

$$(7) \quad 0 \longrightarrow M \xrightarrow{P^T} M' \xrightarrow{U^T} \mathbb{Z}^r \xrightarrow{Q'} K' \xrightarrow{F} K \longrightarrow 0,$$

where M is a sublattice of M' of index $|\det(U)|$ and the map $F : K' \simeq K \times H \rightarrow K$ is the first projection.

It follows that knowing just A , we can form a short exact sequence (6) by finding a suitable submatrix of $A - \mathbf{1}$ of rank n . Then by computing the torsion part of K' we can deduce the map F of (7) and hence the sublattice $M = \ker(F \circ Q')$ of M' . Up to a unimodular basis change, this gives P^T and hence the minimal generators n_1, \dots, n_r of Σ . This is enough to recover X since it is \mathbb{Q} -Fano. Indeed by [40, Prop. 4.3] the cones of the fan Σ are the cones over the faces of $\Delta_2^* = \text{Conv}(n_1, \dots, n_r)$. \square

Let us now turn to the groups of diagonal automorphisms of X that stabilize W . The stabilizer of the monomials of W under the induced action of the big torus $\hat{T} \cong (\mathbb{C}^*)^r$ on the ring $\mathbb{C}[x_1, \dots, x_r]$ is

$$\text{Aut}(W) = \left\{ (\lambda_1, \dots, \lambda_r) \in \hat{T} \mid \prod_{j=1}^r \lambda_j^{a_{ij}} = 1, \text{ for } i = 1, \dots, s \right\}.$$

Let Ψ and $G_\Sigma = \ker(\Psi)$ as defined in section 1. The group $\text{Aut}(W)$ does not contain G_Σ in general, but if we consider the special linear parts $J_\Sigma = G_\Sigma \cap \text{SL}_r(\mathbb{C})$ and

$\mathrm{SL}(W) = \mathrm{Aut}(W) \cap \mathrm{SL}_r(\mathbb{C})$, we have $J_\Sigma \subset \mathrm{SL}(W)$. Indeed, for all $(\lambda_1, \dots, \lambda_r) \in \hat{T} \cap \mathrm{SL}_r(\mathbb{C})$ we have $\prod_{j=1}^r \lambda_j = 1$ and since $a_{ij} = \langle u_i, n_j \rangle + 1$ we have

$$\mathrm{SL}(W) = \{ \lambda \in \hat{T} \cap \mathrm{SL}_r(\mathbb{C}) : \Psi(\lambda)(u) = 1, \forall u \in M_W \},$$

where M_W is the sublattice of M generated by u_1, \dots, u_s . Moreover

$$J_\Sigma = \{ \lambda \in \hat{T} \cap \mathrm{SL}_r(\mathbb{C}) : \Psi(\lambda)(u) = 1, \forall u \in M \}.$$

Note that contrary to the original BHK construction, the group $\mathrm{SL}(W)$ here can be infinite. But when this is the case, then J_Σ is infinite as well and the quotient is finite.

Lemma 4.5. *The homomorphism Ψ defines a perfect pairing*

$$\bar{\Psi} : \mathrm{SL}(W)/J_\Sigma \times M/M_W \rightarrow \mathbb{C}^*, ([\lambda], [u]) \mapsto \Psi(\lambda)(u).$$

In particular $\mathrm{SL}(W)/J_\Sigma$ is isomorphic to M/M_W .

Proof. The homomorphism $\bar{\Psi}$ is clearly well-defined and it induces an injective homomorphism from $\mathrm{SL}(W)/J_\Sigma$ to $\mathrm{Hom}(M/M_W, \mathbb{C}^*)$ by definition of J_Σ . Let (u_1, \dots, u_n) be a basis of M such that there exist positive integers $k_1, \dots, k_n \in \mathbb{N}$ with $(k_1 u_1, \dots, k_n u_n)$ a basis of M_W . We have $M/M_W \simeq \bigoplus_{i=1}^n \mathbb{Z}/k_i \mathbb{Z}$. To prove the surjectivity we will prove that there exists an element of $\mathrm{SL}(W)$ whose image is the homomorphism sending u_1 to $e^{2\pi i/k_1}$ and u_i to 1 for $i > 1$. Consider the dual basis of $(k_1 u_1, \dots, k_n u_n)$. This is a \mathbb{Z} -basis of M_W^\vee , let us denote it by (m_1, \dots, m_n) .

Now pick a r -tuple of rational numbers $w_1, \dots, w_r \in \mathbb{Q}$ such that $\sum_{j=1}^r w_j n_j = m_1$ and write $e^{i2\pi w} = (e^{i2\pi w_1}, \dots, e^{i2\pi w_r})$. Since m_1 is an element of M_W^\vee , we have

$$\sum_{j=1}^r w_j \langle n_j, u \rangle = \langle m_1, u \rangle \in \mathbb{Z} \quad \text{for all } u \in M_W,$$

which implies that $e^{i2\pi w} \in \mathrm{SL}(W)$. It remains to observe that

$$e^{2\pi i \sum_{j=1}^r w_j \langle u_1, n_j \rangle} = e^{2\pi i/k_1}, \quad e^{2i\pi \sum_{j=1}^r w_j \langle u_i, n_j \rangle} = 1 \text{ for } i > 1.$$

A similar argument holds for any u_i , proving the surjectivity. \square

We now choose a subgroup G of $\mathrm{SL}(W)$ containing J_Σ , or equivalently a subgroup $\tilde{G} = G/J_\Sigma$ of $\mathrm{SL}(W)/J_\Sigma$. Observe that by Lemma 4.5 there exists a unique lattice M_G satisfying $M_W \subseteq M_G \subseteq M$ such that

$$(8) \quad G = \left\{ \lambda \in \hat{T} \cap \mathrm{SL}_r(\mathbb{C}) \mid \Psi(\lambda)(u) = 1, \forall u \in M_G \right\}.$$

We thus define an orbifold $Y_{W,G} = Y_W/\tilde{G}$, the family of which we denote by $\mathcal{F}(A, \tilde{G})$.

The following generalizes [42, Lemma 2.3].

Proposition 4.6. *Let X, G be as in Construction 4.3. Then the quotient map $X \rightarrow X/G$ induces the identity $\mathcal{R}(X/G) \rightarrow \mathcal{R}(X)$ and $\mathrm{Cl}(X/G) \cong \mathrm{Cl}(X) \oplus G$.*

Proof. By Lemma 1.1 it is enough to prove that the quotient map is induced by a lattice monomorphism $N \rightarrow N_G$ such that the primitive generators $n_1, \dots, n_r \in N$ of the rays of the fan of X remain primitive in N_G . In fact, assume that $n_i = kn$ for some $n \in N_G$ and some integer k . Since $N_G \subset M_W^\vee$, then $a_{ij} - 1 = \langle n_i, u_j \rangle = k \langle n, u_j \rangle \in k\mathbb{Z}$ for all j . Since a monomial in W can not contain all the variables (otherwise it would correspond to $0 \in M$), this gives $k = 1$. \square

Proposition 4.7. *The orbifolds $Y_{W,G}$ are Calabi-Yau varieties for general $\alpha_i \in \mathbb{C}^*$.*

Proof. By construction Y_W is an anticanonical hypersurface in X . The quotient $Y_W \rightarrow Y_W/G$ is clearly induced by the quotient $X \rightarrow X/G$ induced by the inclusion $N \subseteq N_G = M_G^\vee$. Let $\Delta_2 \subset M_{\mathbb{Q}}$ be the anticanonical polytope of X and Δ_1 be the Newton polytope of Y_W . The anticanonical polytope of X/G is equal to $\Delta_2 \subset (M_G)_{\mathbb{Q}} = M_{\mathbb{Q}}$ by Proposition 4.6. The monomials in the defining equation of Y_W/G are the lattice points $u_i \in M_W \subset M_G$. Thus, the Newton polytope of Y_W/G is clearly Δ_1 . Since (Δ_1, Δ_2) is a good pair in $M_{\mathbb{Q}}$ by Corollary 1.6, then the same holds in $(M_G)_{\mathbb{Q}}$ by Lemma 1.7. By Theorem 1 and Remark 2.3 the orbifold $Y_{W,G}$, whose monomials in Cox coordinates correspond to the vertices of Δ_1 , are Calabi-Yau varieties. \square

Let us now define the dual family $\mathcal{F}(A^T, \tilde{G}^*)$. Let X^* be the toric variety whose fan is the collection of cones over the faces of Δ_1 , considered with respect to the lattice M_W generated by u_1, \dots, u_s . The transposed matrix A^T of A gives the family

$$W^*(y_1, \dots, y_s) = \sum_{i=1}^r \beta_i \prod_{j=1}^s y_j^{a_{ji}}, \quad \beta_i \in \mathbb{C}^*,$$

where y_1, \dots, y_s are the Cox coordinates of X^* . Observe that by Lemma 4.4 the toric variety X^* can be recovered uniquely by the matrix A^T .

We now define the transposed of the group G . If $M^\vee \subseteq M_G^\vee \subseteq M_W^\vee$ denote the dual lattices and $\hat{T}^\vee = (\mathbb{C}^*)^s$ is an algebraic torus of dimension s , we define

$$(9) \quad G^* := \left\{ \nu \in \hat{T}^\vee \cap \mathrm{SL}_s(\mathbb{C}) \mid \Psi^*(\nu)(n) = 1, \text{ for all } n \in M_G^\vee \right\},$$

where

$$\Psi^* : \hat{T}^\vee \rightarrow \mathrm{Hom}(N, \mathbb{C}), \quad \nu \mapsto (n \mapsto \prod_{i=1}^s \nu_i^{\langle u_i, n \rangle}).$$

Let us show that this generalizes the original definition of the transposed group given by Krawitz (5) (see also [12, Proposition 2.3.1]).

Lemma 4.8. *When the good pair (Δ_1, Δ_2) is formed by simplices then the definitions (5) and (9) coincide.*

Proof. The proof relies on the fact that when $r = s = n + 1$ the diagonal groups preserving the polynomials W and W^* are finite, hence formed of $(n + 1)$ -uples of roots of unity. Let us write $e^{2\pi i v} = (e^{2\pi i v_0}, \dots, e^{2\pi i v_n})$ for all $v \in \mathbb{Q}^{n+1}$. Given a subgroup G with $J_\Sigma \subset G \subset \mathrm{SL}(W)$ we will denote by G_1^* its transposed as defined in (5) and G_2^* its transposed as in (9). It is proven in [1, §3.5] that

$$G_1^* := \{e^{2\pi i v} \in \hat{T}^\vee \cap \mathrm{SL}_{n+1}(\mathbb{C}) \mid v^T A w \in \mathbb{Z} \text{ for all } e^{2\pi i w} \in G\}.$$

We first remark that for all $(v, w) \in \mathbb{Q}^{n+1} \times \mathbb{Q}^{n+1}$ we have

$$e^{2\pi i v}, e^{2\pi i w} \in \mathrm{SL}_{n+1}(\mathbb{C}) \Rightarrow \sum_{i,j=0}^n v_i w_j \in \mathbb{Z},$$

and that by (8), for all $w \in \mathbb{Q}^{n+1}$ such that $e^{2\pi i w} \in \mathrm{SL}_{n+1}(\mathbb{C})$ we have

$$e^{2\pi i w} \in G \Leftrightarrow \forall u \in M_G, \sum_{j=0}^n \langle u, n_j \rangle w_j \in \mathbb{Z} \Leftrightarrow \sum_{j=0}^n w_j n_j \in M_G^\vee.$$

It follows that for all $v \in \mathbb{Q}^{n+1}$ such that $e^{2\pi iv} \in \mathrm{SL}_{n+1}(\mathbb{C})$ we have

$$\begin{aligned} e^{2\pi iv} \in G_1^* &\Leftrightarrow \sum_{i,j=0}^n v_i(\langle u_i, n_j \rangle + 1)w_j \in \mathbb{Z} \text{ for all } e^{2\pi iw} \in G \\ &\Leftrightarrow \left\langle \sum_{i=0}^n v_i u_i, \sum_{j=0}^n w_j n_j \right\rangle \in \mathbb{Z} \text{ for all } e^{2\pi iw} \in G \\ &\Leftrightarrow \sum_{i=0}^n v_i u_i \in M_G. \end{aligned}$$

On the other hand, writing $\nu = e^{2\pi iv}$, definition (9) is equivalent to

$$G_2^* := \{e^{2\pi iv} \in \hat{T}^\vee \cap \mathrm{SL}_{n+1}(\mathbb{C}) \mid \sum_{i=0}^n v_i \langle u_i, n \rangle \in \mathbb{Z} \text{ for all } n \in M_G^\vee\}.$$

Since for all $v \in \mathbb{Q}^{n+1}$ we have

$$\sum_{i=0}^n v_i \langle u_i, n \rangle \in \mathbb{Z} \text{ for all } n \in M_G^\vee \Leftrightarrow \sum_{i=0}^n v_i u_i \in M_G,$$

we get $G_1^* = G_2^*$ so that the two definitions are indeed equivalent. \square

Remark 4.9. In [1] it is proved that G^* is the orthogonal complement of G with respect to the bilinear form

$$\mathrm{SL}(W)/J_W \times \mathrm{SL}(W^*)/J_{W^*} \rightarrow \mathbb{C}^*, (e^{2\pi iv}, e^{2\pi iw}) \mapsto e^{2\pi iv^T A w}.$$

It can be easily proved that the same description can be given for the generalized Berglund-Hübsch-Krawitz construction.

Definition 4.10. The *generalized Berglund-Hübsch-Krawitz dual family* of the family $\mathcal{F}(A, \tilde{G})$ is the family $\mathcal{F}(A^T, \tilde{G}^*)$ formed by the orbifolds

$$Y_{W^*, \tilde{G}^*} = \{W^* = 0\}/\tilde{G}^*.$$

We want to emphasize that the above construction is a direct generalization of the original BHK construction, simply replacing canonical simplices (with the quasimoothness condition) by arbitrary canonical polytopes. It is quite striking that the transposition rule extends naturally to the case of a non square matrix, thanks to Cox construction of homogeneous coordinates on the ambient toric varieties.

We conclude with the proof of Theorem 2, which shows that the generalized BHK construction can be described as a duality of good pairs.

Proof of Theorem 2. Let $\Delta_2 \subset M_{\mathbb{Q}}$ be the anticanonical polytope of X and $\Delta_1 \subset \Delta_2$ be the Newton polytope of Y_W . We recall that the anticanonical polytope of X/\tilde{G} is the same polytope $\Delta_2 \subset (M_G)_{\mathbb{Q}}$ and the Newton polytope of $Y_{W,G}$ is $\Delta_1 \subset (M_G)_{\mathbb{Q}}$. Thus (Δ_1, Δ_2) is a good pair in $(M_G)_{\mathbb{Q}}$ as already explained in the proof of Proposition 4.7. We now consider the polar pair (Δ_2^*, Δ_1^*) as a good pair in $(M_W^\vee)_{\mathbb{Q}}$. By construction X^* is defined by the fan over the faces of Δ_1 and the vertices of Δ_2^* correspond to the monomials of W^* . Thus Δ_1^* is the anticanonical polytope of X^* and Δ_2^* is the Newton polytope of Y_{W^*} . Looking at the definition of the transposed group G^* one immediately sees that the pair (Δ_2^*, Δ_1^*) seen in $(M_G^\vee)_{\mathbb{Q}}$ defines the toric variety X^*/\tilde{G}^* and the hypersurface Y_{W^*, G^*} . \square

Example 4.11. Let $X = \mathbb{P}(5, 5, 4, 4, 2)$ and let $\Delta_1 \subset M_{\mathbb{Q}}$ be the convex hull of the lattice points $u_1, \dots, u_5 \in M$ corresponding to the monomials

$$x_1^3x_2, x_2^4, x_3^5, x_4^5, x_5^{10}.$$

The polytope Δ_1 is a simplex. We denote by $\Delta_2 \subset M_{\mathbb{Q}}$ the anticanonical polytope of X , thus (Δ_1, Δ_2) is a good pair and they both are reflexive. Since Δ_1 and Δ_2 are simplices, then the diagonal group of automorphisms preserving

$$W = x_1^3x_2 + x_2^4 + x_3^5 + x_4^5 + x_5^{10}$$

is finite and $\mathrm{SL}(W)/J_W \cong \mathbb{Z}/5\mathbb{Z}$. The polynomial W is quasismooth since it satisfies the conditions of [34, Theorem 1]. The BHK construction gives a dual Calabi-Yau hypersurface in the toric variety Y which is the quotient of $\mathbb{P}(10, 6, 6, 5, 3)$ by the group of order five generated by $g = (1, e^{2\pi i \frac{1}{5}}, e^{2\pi i \frac{4}{5}}, 1, 1)$. The hypersurface is defined by the following polynomial in the Cox coordinates of Y :

$$W^* = y_1^3 + y_1y_4^4 + y_2^5 + y_3^5 + y_5^{10}.$$

The threefold defined by W in X is a birational projective model for the Borcea-Voisin Calabi-Yau threefold obtained from the elliptic curve $E := \{x_0^2 + x_1^3x_2 + x_2^4 = 0\} \subset \mathbb{P}(2, 1, 1)$ and the K3 surface $S := \{y_0^2 + y_1^5 + y_2^5 + y_3^{10} = 0\} \subset \mathbb{P}(5, 2, 2, 1)$ (see [2, Proposition 4.4]).

Example 4.12. Let $X = \mathbb{P}(5, 5, 4, 4, 2)$ as in Example 4.11 and let $\Delta'_1 \subset \Delta_2$ be the convex hull of the lattice points $u_1, \dots, u_8 \in M$ corresponding to the monomials

$$x_1^3x_2, x_2^4, x_3^5, x_4^5, x_5^{10}, x_1^2x_5^5, x_1^2x_4^2x_5, x_1^2x_3^2x_5.$$

Observe that $\Delta_1 \subset \Delta'_1$ and Δ'_1 is not a simplex. We have that (Δ'_1, Δ_2) is a good pair of reflexive polytopes and one can prove that the general element of $\mathcal{F}_{\Delta'_1, \Delta_2^*}$ is quasismooth. The lattice generated by u_1, \dots, u_8 is equal to M so that the only possible choice for \tilde{G} is the trivial group. Thus starting from Δ'_1 one obtains the family of Calabi-Yau hypersurfaces $\mathcal{F}(A', G)$

$$\alpha_1x_1^3x_2 + \alpha_2x_2^4 + \alpha_3x_3^5 + \alpha_4x_4^5 + \alpha_5x_5^{10} + \alpha_6x_1^2x_5^5 + \alpha_7x_1^2x_4^2x_5 + \alpha_8x_1^2x_3^2x_5 = 0.$$

The generalized BHK construction gives the dual family $\mathcal{F}((A')^T, G^*)$

$$\beta_1y_1^3y_3^2y_4^2y_5^2 + \beta_2y_1y_2^4 + \beta_3y_3^5y_4y_5y_6^{10} + \beta_4y_4^2y_7^5 + \beta_5y_5^2y_8^5 = 0$$

in the toric variety whose Cox ring is $\mathbb{C}[y_1, \dots, y_8]$ with Class group isomorphic to \mathbb{Z}^4 and grading matrix

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 5 & 0 & 0 & 10 & 1 & 4 & 0 \\ 0 & 5 & 0 & 5 & 5 & 1 & 2 & 2 \\ 0 & 7 & 1 & 4 & 9 & 1 & 4 & 2 \end{pmatrix}.$$

5. ABOUT THE STRINGY HODGE NUMBERS OF GOOD PAIRS

The polar duality between good pairs provides a duality between families of Calabi-Yau hypersurfaces of \mathbb{Q} -Fano toric varieties. Of course a natural question arises:

Question. Do the families $\mathcal{F}_{\Delta_1, \Delta_2^*}$ and $\mathcal{F}_{\Delta_2^*, \Delta_1}$ associated to a good pair (Δ_1, Δ_2) satisfy the topological mirror test, i.e.

$$h_{st}^{p,q}(X) = h_{st}^{n-p,q}(X^*), \quad 0 \leq p, q \leq n$$

for general $X \in \mathcal{F}_{\Delta_1, \Delta_2^*}$ and $X^* \in \mathcal{F}_{\Delta_2^*, \Delta_1}$?

This is known to be true for anticanonical hypersurfaces of toric Fano varieties (i.e. when $\Delta_1 = \Delta_2$) by Batyrev and Borisov [8] and in the classical Berglund-Hübsch case by Chiodo and Ruan [17] (i.e. when Δ_1, Δ_2 are simplices and quasimoothness holds) in terms of Chen-Ruan orbifold cohomology. Observe that in the latter case orbifold Hodge numbers are known to be the same as the stringy Hodge numbers (see [44] and [13]). Unfortunately we are unable to give a complete answer at the moment. Example 5.3 shows that the answer is negative in general and Example 5.4 leads us to think that the answer could be positive if quasimoothness holds.

We recall that in case the polytope is reflexive of dimension four Hodge numbers can be computed in terms of combinatorial properties of the polytope [5, Corollary 4.5.1].

The following result allows to compute the stringy Hodge numbers for good pairs of four dimensional reflexive polytopes. We recall that, given a reflexive polytope Δ , an *MPCP resolution* of the toric variety X_Δ is defined by a complete, simplicial projective fan which is a subdivision of the normal fan Σ_Δ to Δ and whose rays are generated by nonzero lattice points of Δ^* . In [24] the author introduces the more general notion of Δ -maximal fan, which has the same definition of an MPCP resolution except for the fact that it is not necessarily projective and it does not need to refine Σ_Δ . By [5, Corollary 4.2.3] an MPCP resolution induces a crepant resolution of the general anticanonical hypersurface of X_Δ in case Δ is four dimensional. By [24, Theorem 4.9], given any Δ -maximal fan Σ for a four dimensional reflexive polytope Δ , the general member of the anticanonical linear series of the toric variety X_Σ is smooth.

Proposition 5.1. *Let $\Delta_1 \subseteq \Delta_2$ be four dimensional reflexive polytopes. The general elements of the families $\mathcal{F}_{\Delta_1, \Delta_2^*}$ and $\mathcal{F}_{\Delta_2^*, \Delta_1}$ have the same Hodge numbers.*

Proof. Let Σ_2 be the normal fan to Δ_2 and Σ'_2 be the fan of an MPCP resolution of X_{Δ_2} . By [24, Lemma 6.1] there exists a projective Δ_1 -maximal fan $\tilde{\Sigma}_2$ which refines Σ'_2 , and thus Σ_2 . Let $\phi: X_{\tilde{\Sigma}_2} \rightarrow X_{\Sigma_2}$ be the corresponding toric morphism. By [24, Theorem 4.9] the general member \tilde{Y} of the anticanonical linear series of $X_{\tilde{\Sigma}_2}$ is smooth. Moreover, by Corollary 2.7, \tilde{Y} is a crepant resolution of the general member Y of $\mathcal{F}(\Delta_1)$. Now let \tilde{Z} be a crepant resolution of the general anticanonical hypersurface Z of X_{Δ_1} , induced by an MPCP resolution. By [6, Theorem 3.12] \tilde{Y} and Y have the same stringy Hodge numbers, and the same holds for \tilde{Z} and Z . Since \tilde{Y} and \tilde{Z} are birational smooth Calabi-Yau threefolds (since they are given by the same Newton polytope), then they have the same Hodge numbers [7, Theorem 1.1], which achieves the proof. \square

Corollary 5.2. *Let (Δ_1, Δ_2) be a good pair of reflexive, four dimensional polytopes. The associated families $\mathcal{F}_{\Delta_1, \Delta_2^*}$ and $\mathcal{F}_{\Delta_2^*, \Delta_1}$ satisfy the topological mirror test if and only if the general elements of the families $\mathcal{F}_{\Delta_1, \Delta_1^*}$ and $\mathcal{F}_{\Delta_2, \Delta_2^*}$ have the same Hodge numbers.*

Proof. It follows from Proposition 5.1 and [5, Corollary 4.5.1], which gives that $h^{1,1}(\Delta_2) = h^{2,1}(\Delta_2^*)$ and viceversa. \square

Example 5.3. Consider the following four dimensional reflexive polytopes:

$$\begin{aligned}\Delta_1 &= \text{Conv}((1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), (-1,-1,-1,-1)), \\ \Delta_2 &= \text{Conv}((1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), \\ &\quad (-1,-1,-1,-1), (1,1,1,1), (0,0,0,-1)).\end{aligned}$$

By Proposition 5.1 and [5, Corollary 4.5.1] the Hodge numbers of (Δ_1, Δ_2) are $(h^{1,1}, h^{2,1}) = (101, 1)$ (observe that the variety associated to the normal fan to Δ_1 is \mathbb{P}^4), whereas the Hodge numbers of the dual pair are $(3, 79)$. Thus the dual families $\mathcal{F}_{\Delta_1, \Delta_2^*}$ and $\mathcal{F}_{\Delta_2^*, \Delta_1}$ do not satisfy the topological mirror test. A direct computation shows the general member of $\mathcal{F}_{\Delta_1, \Delta_2^*}$ is not quasismooth. This example appears in [23, §3].

Example 5.4. Let $\mathcal{F}_{\Delta'_1, \Delta_2^*}$ be the family of hypersurfaces of $\mathbb{P}(5, 5, 4, 4, 2)$ associated to the reflexive polytope $\Delta'_1 \subset \Delta_2$ defined in Example 4.12. The Hodge numbers of Δ'_1 and Δ_2 are equal to $(h^{1,1}, h^{2,1}) = (15, 39)$ by Batyrev formulas. Thus by Corollary 5.2 the dual families $\mathcal{F}_{\Delta'_1, \Delta_2^*}$ and $\mathcal{F}_{\Delta_2^*, \Delta'_1}$ satisfy the topological mirror test.

Combinatorially, the duality of good pairs can be seen as included in the unified setting for mirror symmetry sketched in [12, §7], except for the regularity condition described in [12, Proposition 7.1.3]. We intend to explore such condition, in terms of combinatorial properties of polytopes, in a further work.

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